A periodic boundary-layer flow in magnetohydrodynamics

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It is the purpose of this paper to solve a boundary-value problem posed by induction electromagnetic pumps and generators. Solutions are obtained by an expansion technique and a momentum method for the laminar, incompressible flow problem. For large values of the interaction parameter ($\mu^2\sigma H_0^2\lambda/\rho u_e$) viscous effects are shown to be restricted to periodic boundary layers. In regions of high-field strength a local Hartmann solution is valid. Where the applied field is weak an inertial boundary layer is present which thickens in the upstream direction. A logical explanation of this phenomenon is given. The condition that a boundary-layer type flow exist is obtained and is shown to be in general satisfied. The results show that inviscid theory may be used to calculate the overall performance of electromagnetic pumps and generators while the boundary-layer theory developed here may be used to obtain the wall shear stress.

1. Introduction

Boundary-value problems solved in fluid mechanics usually arise from actual applications. Examples are Poiseuille flow and the boundary layer on a flat plate. Similarly, the important solutions in magnetohydrodynamics are usually related to applications. One of the best examples is Hartmann flow (see Hartmann 1937), which finds an application in the crossed-fields electromagnetic conduction pump used in nuclear reactors. The Hartmann solution determines the velocity profile for the laminar flow of a viscous, incompressible fluid in the presence of uniform electric and magnetic fields.

While several designs of Hartmann-type pumps are in use, inductance-type electromagnetic pumps offer stiff competition. The induction-type pump utilizes a moving magnetic field which essentially drags the conducting liquid with it, eliminating the necessity for electrodes. The moving magnetic field may be generated by alternating current in stationary coils or a steady current in moving coils. A thorough discussion of the advantages of the induction-type pump has been given by Blake (1959).

It is the purpose of this paper to solve a boundary-value problem posed by induction electromagnetic pumps. Since induction designs are also being considered in magnetohydrodynamic power generation (see Bernstein *et al.* 1961), the results obtained here should also be applicable in this field. The performance of induction-type pumps has been studied by Blake (1957), but this author did not consider viscous effects. Harris (1960) discussed qualitatively the viscous effects, but did not attempt a solution of the basic equations.

2. Formulation of the problem

The idealized problem is a two-dimensional, laminar flow of an incompressible, electrically conducting fluid between parallel walls. The magnetic field is prescribed and is of the form

$$H_y = H_0 \sin\left[2\pi\left(\frac{x_1 - u_f t}{\lambda}\right)\right], \quad H_x = 0. \tag{1}$$

The field is uniform across the channel and moves down the channel with a velocity u_f. The problem is illustrated in figure 1. Prescribing the field as in equation (1) implies neglect of induced fields and of fringing of the applied field.

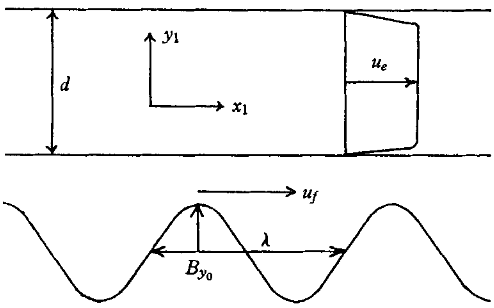

FIGURE 1. Formulation of the problem in laboratory co-ordinates.

Induced fields may be neglected if the magnetic Reynolds number ($\mathscr{R}_m = \mu\sigma u\lambda$) is sufficiently small. For all devices using liquid metals the appropriate magnetic Reynolds number is indeed small, usually about 0·1 to 0·01. If induced effects are negligible, the fringing of the applied field is obtained by solving Laplace's equation in the region between the walls of the channel. The assumption of a field which is uniform across the channel is appropriate if the ratio of channel width (d) to the wavelength of the applied field (λ) is small. Again this requirement is satisfied in actual devices. A detailed quantitative consideration of these approximations has been given by Lyons & Turcotte (1962).

Since the magnetic field is prescribed, only the momentum and continuity equations, along with an Ohm's law, need be considered in obtaining a solution. The equations are written in a reference frame moving at a velocity u_f with respect to the laboratory frame. With $x' = x_1 - u_f t$, equation (1) may be rewritten

$$H_y = H_0 \sin(2\pi x'/\lambda), \quad H_x = 0. \tag{2}$$

Since in this reference frame the velocity boundary conditions are also independent of time, a steady solution periodic in x' might be expected. Some discussion of the pressure is necessary here. One of the boundary conditions in the laboratory co-ordinates for an actual device is that the input pressure at $x = x_0$ be given. Then the pressure increase through the pump is the output. Therefore the pressure in the (x', t') co-ordinate system is in fact a function of time t'. However the time derivative of the pressure does not enter the governing equations, only the gradient, so a steady solution periodic in x' is in fact appropriate. However, in applying the results it is necessary to note that the pressure level in the (x', t')

co-ordinate system is in fact a function of t'. The appropriate equations may now be written

$$\nabla . \mathbf{u}' = 0, \tag{3}$$

$$\mathbf{u}' . \nabla \mathbf{u}' = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}' + \frac{\mu}{\rho}(\mathbf{j} \times \mathbf{H}), \tag{4}$$

$$\mathbf{j} = \sigma[\mathbf{E} + \mu(\mathbf{u}' \times \mathbf{H})], \tag{5}$$

where M.K.S. units are used. The electric-field vector $\mathbf{E}$ is taken to be zero since no applied electric field is present and no induced electric field occurs in the steady problem considered. Such an assumption is appropriate if either (1) the induced currents close on themselves as in an annular geometry, or (2) the wire (or wires) closing the current paths has negligible resistivity and moves with a velocity u_f with respect to the laboratory frame of reference (see Panofsky & Phillips 1955). The first condition is satisfied in most actual designs. The current flows only in the z-direction.

When equations (2) and (5) are substituted into equation (4) with equation (3) the following component equations are obtained:

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \tag{6}$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{1}{\rho}\frac{\partial p}{\partial x'} + \nu\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) - \frac{\mu^2\sigma}{\rho} H_0^2 u' \sin^2\left(\frac{2\pi x'}{\lambda}\right), \tag{7}$$

$$u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y'} = -\frac{1}{\rho}\frac{\partial p}{\partial y'} + \nu\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right), \tag{8}$$

and the boundary conditions may be written

$$u' = u_f \quad \text{at} \quad y' = 0, d;$$

$$v' = 0 \quad \text{at} \quad y' = 0, d;$$

and

$$\int_0^d u' \, dy' = \text{const.}$$

The problem as posed is illustrated in figure 2. It will now be postulated that viscous effects are restricted to boundary-layer regions near the walls of the channel. The conditions under which such a hypothesis is valid will be discussed when solutions are obtained. As in other boundary-layer problems the inviscid solution valid in the core of the channel may be matched to the boundary-layer solutions valid near the walls. Also the equations may be simplified according to the methods of Prandtl (see, for example, Schlichting 1955). The appropriate boundary-layer approximations reduce equations (7) and (8) to

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{1}{\rho}\frac{\partial p}{\partial x'} + \nu\frac{\partial^2 u'}{\partial y'^2} - \frac{\mu^2\sigma}{\rho} H_0^2 u' \sin^2\left(\frac{2\pi x'}{\lambda}\right), \tag{9}$$

$$0 = \frac{\partial p}{\partial y}. \tag{10}$$

The inviscid-core solution, as obtained by Blake (1957) and Harris (1960), may be written

$$u' = -u_f + u_e, \quad v' = 0,$$

$$\frac{dp}{dx'} = \mu^2 \sigma H_0^2 (u_f - u_e) \sin^2\left(\frac{2\pi x'}{\lambda}\right). \tag{11}$$

This is uniform flow with constant velocity u_e in the laboratory reference frame, and static pressure varying with x_1 and t. As previously noted the static pressure varies with both x' and t' in order to satisfy the inlet condition; however the

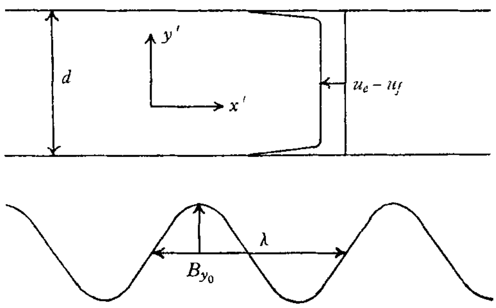

FIGURE 2. Formulation of the problem in field-fixed co-ordinates.

pressure gradient is a function only of x'. When equation (11) is combined with equations (9) and (10), the boundary-layer problem is formulated; viz.

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = \frac{\mu^2\sigma}{\rho} H_0^2 (u_e - u_f - u')\sin^2\left(\frac{2\pi x'}{\lambda}\right) + \nu\frac{\partial^2 u'}{\partial y'^2}, \tag{12}$$

along with equation (6) and the boundary conditions

$$u' = -u_f, \quad v' = 0 \quad \text{at} \quad y' = 0;$$

$$u' = -u_f + u_e \quad \text{at} \quad y' = \infty.$$

Since the boundary layer is assumed to be thin compared with the width of the channel, the conditions at the outer edge of the boundary layer are applied at $y = \infty$ in boundary-layer co-ordinates. Note that the body-force term goes to zero at the outer edge of the boundary layer where the pressure gradient is balanced by the magnetic body force, and the body-force term always acts to reduce the velocity deficit within the boundary layer. For this reason a periodic solution can be expected.

In order to obtain further insight into the problem, dimensionless variables and parameters are introduced. Let

$$x = x'/\lambda, \quad y = y'/\lambda, \quad u = u'/u_e,$$

$$S = \frac{u_f - u_e}{u_e}, \quad Q = \frac{\mu^2\sigma H_0^2 \lambda}{\rho u_e}, \quad R_e = \frac{u_e \lambda}{\nu}.$$

Equations (6) and (12) may be written in terms of these variables as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{13}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -Q(S+u)\sin^2 2\pi x + \frac{1}{R_e}\frac{\partial^2 u}{\partial y^2}. \tag{14}$$

The interpretation of the viscous Reynolds number (R_e) is well known. The interaction parameter (Q) represents the ratio of the pondermotive force to the inertia force. The square of the Hartmann number ($M^2 = R_e Q$) represents the ratio of the pondermotive force to the viscous force. The Hartmann number defined above differs from the usual form in that it is based on wavelength rather than channel width.

If Q is assumed to be large the inertia terms on the left side of equation (14) should be negligible as long as the forcing term on the right side is large. Thus a balance between the body force and the viscous term should be appropriate except where the applied magnetic field nearly vanishes. It will be seen that this balance leads to a Hartmann-type boundary-layer solution (see Hartmann 1937) valid locally. The inertia terms become important only where the interaction term becomes small. The inertia terms retard the growth of the Hartmann-type boundary layer in the region where the interaction term is small. For all liquids of interest the viscosity is small so that the viscous Reynolds number and Hartmann number are large. Also, high-performance induction devices require large interaction parameters. For the relatively low-performance induction pump tested by Blake (1957), for example, $Q = 2{\cdot}75$, $R_e = 4{\cdot}72\times10^6$ and $M = 3{\cdot}61\times10^3$. A series solution valid for large Q will be obtained first. Then a momentum method will be applied to obtain a solution valid for Q of order one.

3. Power-series solution

Since equation (14) is a non-linear partial differential equation, it is not possible to obtain a general solution. Instead a series solution in powers of $1/Q$, valid for large Q, will be found. If a solution of the form

$$u = u_0 + Q^{-1}u_1 + \dots, \quad v = v_0 + Q^{-1}v_1 + \dots \tag{15}$$

is substituted into equations (13) and (14), the equations for u_0, u_1, v_0 and v_1 may be written

$$O = -M^2(S+u_0)\sin^2 2\pi x + \frac{\partial^2 u_0}{\partial y^2}, \tag{16}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0, \tag{17}$$

$$u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} = -(\sin^2 2\pi x)\, u_1 + \frac{1}{M^2}\frac{\partial^2 u_1}{\partial y^2}, \tag{18}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0. \tag{19}$$

The solution of equations (16) and (17) which satisfies the required boundary conditions ($u_0 = -1-S$, $v_0 = 0$ at $y = 0$; $u_0 = -S$ at $y = \infty$) is

$$u_0 = -\exp\{-My\,|\sin 2\pi x|\} - S, \tag{20}$$

$$v_0 = \frac{2\pi}{M}\cot 2\pi x[My\exp\{-My\,|\sin 2\pi x|\} - |\csc 2\pi x|\,(\exp\{-My\,|\sin 2\pi x|\} - 1)]. \tag{21}$$

This is a Hartmann-type flow where the local value of the Hartmann number $M_L = M\,|\sin 2\pi x|$ determines the velocity profile. In regions where the local

Hartmann number is large the profile has a boundary-layer character. The boundary layer has its minimum thickness where the applied magnetic field is a maximum, and according to this solution the thickness goes to infinity when the magnetic-field strength goes to zero. The boundary-layer thickness is small compared with the channel width only if $\mu H d \sqrt{(\mu/\sigma)} \gg 1$, so that the local Hartmann number based on channel width must be large.

The solution for the first-order correction to this local Hartmann flow will now be found. It is clear from the form of equation (18) that for $M \gg 1$ with Q large the correction term u_1 will be appreciable only when $1/\sin^2 2\pi x$ is large. As a result, if Q is sufficiently large the correction to the zero-order solution is appreciable only where $|\sin 2\pi x| \ll 1$. Therefore the approximation $\sin 2\pi x = 2\pi x$ is made both in equation (18) and in equations (20) and (21) when substituted into equation (18). The fact that the solution obtained is in error for $|\sin 2\pi x|$ of order one does not matter since both the solution obtained and the actual solution are negligibly small in this region under the conditions stated. The resulting equation for u_1 may be written

$$\frac{1}{x} e^{\mp 4\pi Mxy} + \left[\mp 2\pi y SM - \frac{1}{x} \right] e^{\mp 2\pi Mxy} = -(2\pi x)^2 u_1 + \frac{1}{M^2} \frac{d^2 u_1}{dy^2}, \tag{22}$$

where the upper sign is applicable for $x > 0$ and the lower for $x < 0$. The solution to equation (22) is obtained by using the standard methods for first-order, inhomogeneous, linear differential equations. The solution that satisfies the required boundary conditions ($u_1 = 0$ at $y = 0$ and ∞) may be written

$$u_1 = \frac{\pi}{\xi^3} [\tfrac{2}{3} e^{\mp 2\eta} + (-\tfrac{2}{3} \pm \{1 + \tfrac{1}{2}S\} \eta + \tfrac{1}{2} S\eta^2) e^{\mp \eta}], \tag{23}$$

where $\xi = 2\pi x$ and $\eta = 2\pi x M y$. If desired, v_1 may be obtained from equation (19). Equations (20) and (23) are combined with equation (15) to give the x-component of velocity correct to first-order in $1/Q$,

$$u = -(e^{\mp \eta} + S) + \frac{\pi}{Q\xi^3} [\tfrac{2}{3} e^{\mp 2\eta} + (-\tfrac{2}{3} \pm \{1 + \tfrac{1}{2}S\} \eta + \tfrac{1}{2} S\eta^2) e^{\mp \eta}]. \tag{24}$$

From the above expression it is clear that we have really obtained a series solution in inverse powers of the parameter $Q(2\pi x)^3$. That is, equation (24) is an asymptotic expansion valid away from the region near $x = 0$. The effects of inertia on the local Hartmann flow are appreciable, for large Q, only when $[Q(2\pi x)^3]^{-1}$ is of order one.

Before interpreting this result the dimensionless displacement thickness of the boundary layer

$$\left(\delta^* = \int_0^\infty [-S - u] \, dy \right)$$

is found from equation (24). This may be written

$$\delta^* \sqrt{R_e} = \frac{1}{Q^{\frac{1}{2}} |\sin 2\pi x|} \mp \frac{\pi(\frac{2}{3} + \frac{1}{2}S)}{Q^{\frac{3}{2}} (2\pi x)^4}, \tag{25}$$

where again the upper sign applies for $x > 0$, i.e. in the direction of flow in laboratory co-ordinates, and the lower sign for $x < 0$. The zero-order displacement thickness, the first term in equation (25), and the zero-order thickness with the first-order correction are plotted against position in figure 3 for $Q = 100$ and $S = 0{\cdot}5$. This plot clearly demonstrates that the local Hartmann flow is valid

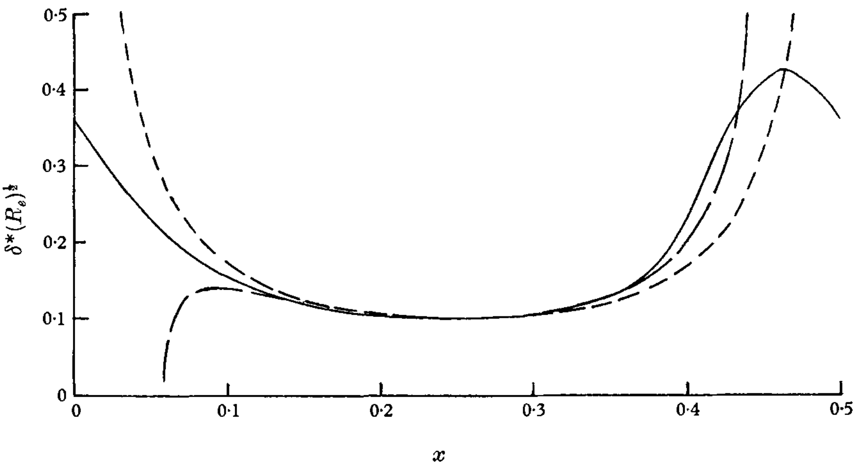

FIGURE 3. Dependence of the dimensionless displacement thickness on position. $Q = 100$, $S = 0{\cdot}5$. – – –, Zero order; — —, corrected to first order; ——, momentum method.

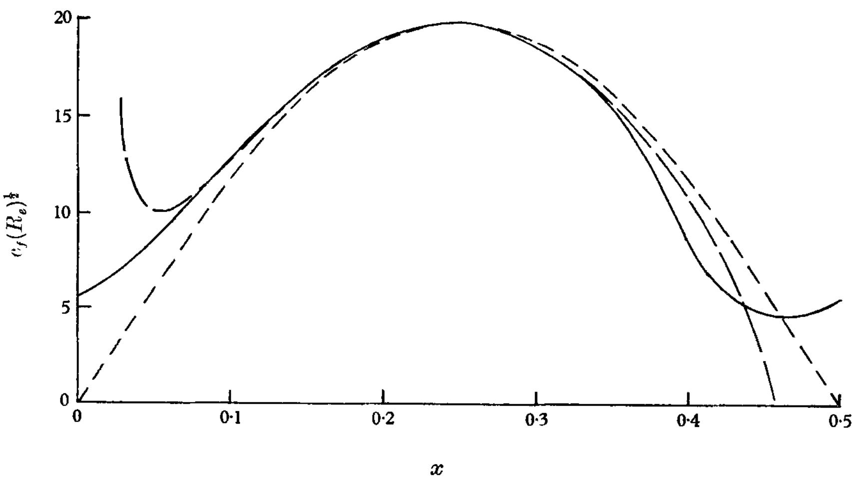

FIGURE 4. Dependence of the local skin-friction coefficient on position. $Q = 100$, $S = 0{\cdot}5$. – – –, Zero order; — —, corrected to first order; ——, momentum method.

away from the nodal points where the applied magnetic field approaches zero. This result would seem to justify the initial hypothesis that a periodic boundary-layer solution is appropriate.

It is interesting to note that the first-order perturbation thickens the boundary layer upstream of the node while the boundary layer is thinner downstream of the node. At first this may seem surprising. However, it must be remembered that in the moving co-ordinate system, in which the steady solution is found, the walls are translating in the negative x-direction with a velocity u_f while the fluid core is translating in the same direction with a velocity $u_f - u_e$. Thus, in the steady-flow co-ordinate system the boundary layer actually represents a flow in the negative x-direction; therefore it is not surprising to find the diffusion of vorticity to be greater upstream of the node in laboratory co-ordinates.

The wall shear stress (τ_w) may also be obtained from equation (24); the local skin-friction coefficient ($c_f \equiv \tau_w/\frac{1}{2}\rho u_e^2$) is found to be

$$c_f\sqrt{R_e} = 2Q^{\frac{1}{2}}\,|\sin 2\pi x| \pm \frac{(\frac{1}{3}+\frac{1}{2}S)}{4\pi x^2 Q^{\frac{1}{2}}}. \tag{26}$$

The dependence of the skin-friction coefficient on position is given in figure 4 for $Q = 100$ and $S = 0{\cdot}5$; the zero-order solution and the solution with first-order correction are included. The skin-friction behaviour is consistent with the displacement-thickness results discussed above. In order to verify further the type of solution postulated a momentum method will now be applied.

4. Momentum method

Momentum-integral methods have often been used to obtain approximate solutions to boundary-layer problems (see Schlichting 1955). Probably the best known example is the solution given by von Kármán and Pohlhausen for the general problem of an incompressible, two-dimensional boundary layer with a pressure gradient. The required momentum equation for the present problem is obtained by integrating equation (14) from $y = 0$ to $y = \infty$. The result, using equation (13), may be written

$$\frac{d}{dx}\left[\int_0^\infty u(S+u)\,dy\right] = -Q(\sin^2 2\pi x)\int_0^\infty (S+u)\,dy - \left|\frac{1}{R_e}\frac{\partial u}{\partial y}\right|_{y=0}. \tag{27}$$

The essence of the momentum method consists of assuming an appropriate relation for the velocity profile $u(y)$ in the boundary layer, such that it satisfies the boundary conditions on u while retaining a free parameter, usually a boundary-layer thickness, to be determined from equation (27).

In the present problem an exponential dependence of velocity on position is chosen, i.e.

$$u = -S - e^{-y/\delta^*}. \tag{28}$$

This functional form satisfies the required boundary conditions and reduces to the exact solution of the equations when a local Hartmann solution is appropriate. While equation (28) forces a similar profile, this restriction should not lead to serious error, since the periodicity of the present problem reduces the accumulation of error. An improved velocity profile would have two free parameters to be determined by the method of von Kármán and Pohlhausen. However, the complexity of the two-parameter form in the present problem is so great that its use is of doubtful value. The parameter $\delta^*(x)$ in equation (28) is identical with the displacement thickness previously introduced.

When equation (28) is substituted into equation (29), a first-order differential equation for the dimensionless displacement thickness is obtained:

$$\frac{d\delta^{*2}}{dx} = 2\frac{Q}{S+\frac{1}{2}}(\sin^2 2\pi x)\,\delta^{*2} - \frac{2}{R_e(S+\frac{1}{2})}. \tag{29}$$

The solution of this equation may be written

$$\frac{R_e(1+2S)}{4}\delta^{*2} = e^{KC}\left[\frac{1}{1-e^{-\frac{1}{2}K}}\int_0^{\frac{1}{2}} e^{-KC}\,dx - \int_0^x e^{-KC}\,dx\right], \tag{30}$$

where
$$K = \frac{2Q}{1+2S},$$

$$C = x - \frac{\sin 4\pi x}{4\pi}, \quad x > 0.$$

The arbitrary constant of integration has been evaluated so that a periodic solution is obtained. An examination of equation (29) shows that the solutions approach periodicity for arbitrary initial conditions; therefore the value of the constant used in equation (30) is appropriate. In figure 3 the displacement thickness given by the momentum method is compared with the values obtained from the local Hartmann solution and the expansion procedure. Agreement between the momentum method and the local Hartmann solution is excellent where the M.H.D. interaction is strong. The initial deviations given by the expansion technique are also in good agreement with the deviations predicted by the integral method. The overall thickness of the boundary layer may now be estimated. The integral solution shows that the effect of the inertial terms is to retard the growth of the boundary layer in the areas of weak interaction. Therefore an estimate for the maximum thickness of the boundary layer may be obtained by taking the thickness of the Hartmann or zero-order solution at the point where the first-order correction previously obtained is of order one. The estimated maximum displacement thickness obtained in this manner is

$$\delta^*(R_e)^{\frac{1}{2}} = Q^{-\frac{1}{6}} \tag{31}$$

and for the conditions considered in figure 3, equation (31) gives $\delta^*(R_e)^{\frac{1}{2}} = 0{\cdot}45$, which is in good agreement with the momentum method. Clearly for the boundary-layer theory to be valid the displacement thickness given in equation (31) must be small compared with the channel width. This requirement may be reduced to the condition $R_e^{\frac{1}{2}}Q^{\frac{1}{6}}\,d/\lambda \gg 1$. Because of the large values of the viscous Reynolds number in actual devices this condition is almost always satisfied and viscous effects are in fact restricted to boundary layers. For example in the pump tested by Blake (1957), $R_e^{\frac{1}{2}}Q^{\frac{1}{6}}\,d/\lambda = 200$.

The local skin-friction coefficient is related to the displacement thickness for the assumed velocity profile according to the relation

$$c_f(R_e)^{\frac{1}{2}} = 2/\delta^*(R_e)^{\frac{1}{2}}. \tag{32}$$

The local skin friction given by the momentum method is compared with the values obtained from the local Hartmann solution and the expansion procedure in figure 4. The discussion of the displacement thickness given above is also applicable for the shear stress.

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